

# SHUBIN REGULARITY FOR THE RADially SYMMETRIC SPATIALLY HOMOGENEOUS BOLTZMANN EQUATION WITH DEBYE-YUKAWA POTENTIAL

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**ABSTRACT.** In this work, we study the Cauchy problem for the radially symmetric spatially homogeneous Boltzmann equation with Debye-Yukawa potential. We prove that this Cauchy problem enjoys the same smoothing effect as the Cauchy problem defined by the evolution equation associated to a fractional logarithmic harmonic oscillator. To be specific, we can prove the solution of the Cauchy problem belongs to Shubin spaces.

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## 1. INTRODUCTION

In this work, we consider the spatially homogeneous Boltzmann equation

$$(1.1) \quad \frac{\partial f}{\partial t} = Q(f, f)$$

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where  $f = f(t, v)$  is the density distribution function depending only on two variables  $t \geq 0$  and  $v \in \mathbb{R}^3$ . The Boltzmann bilinear collision operator is given by

$$Q(g, f)(v) = \int_{\mathbb{R}^3} \int_{S^2} B(v - v_*, \sigma) (g(v'_*)f(v') - g(v_*)f(v)) dv_* d\sigma,$$

where for  $\sigma \in S^2$ , the symbols  $v'_*$  and  $v'$  are abbreviations for the expressions,

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma,$$

which are obtained in such a way that collision preserves momentum and kinetic energy, namely

$$v'_* + v' = v + v_*, \quad |v'_*|^2 + |v'|^2 = |v|^2 + |v_*|^2.$$

For monatomic gas, the collision cross section  $B(v - v_*, \sigma)$  is a non-negative function which depends only on  $|v - v_*|$  and  $\cos \theta$  which is defined through the scalar product in  $\mathbb{R}^3$  by

$$\cos \theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma.$$

Without loss of generality, we may assume that  $B(v - v_*, \sigma)$  is supported on the set  $\cos \theta \geq 0$ , i.e. where  $0 \leq \theta \leq \frac{\pi}{2}$ . See for example [14], [29] for more explanations about the support of  $\theta$ . For physical models, the collision cross section usually takes the form

$$B(v - v_*, \sigma) = \Phi(|v - v_*|)b(\cos \theta),$$

with a kinetic factor

$$\Phi(|v - v_*|) = |v - v_*|^\gamma, \quad \gamma \in ]-3, +\infty[.$$

The molecules are said to be Maxwellian when the parameter  $\gamma = 0$ .

Except for the hard sphere model, the function  $b(\cos \theta)$  has a singularity at  $\theta = 0$ . For instance, in the important model case of the inverse-power potentials,

$$U(\rho) = \frac{1}{\rho^r}, \quad \text{with } r > 1,$$

with  $\rho$  being the distance between two interacting particles in the physical 3-dimensional space  $\mathbb{R}^3$ ,

$$b(\cos \theta) \sin \theta \sim K \theta^{-1-\frac{2}{r}}, \quad \text{as } \theta \rightarrow 0^+.$$

The notation  $a \sim b$  means that there exist positive constants  $C_2 > C_1 > 0$ , such that

$$C_1 a \leq b \leq C_2 a.$$

Notice that the Boltzmann collision operator is not well defined for the case when  $r = 1$  corresponding to the Coulomb potential.

If the inter-molecule potential satisfies the Debye-Yukawa type potentials where the potential function is given by

$$U(\rho) = \frac{1}{\rho e^{\rho^s}}, \quad \text{with } s > 0,$$

the collision cross section has a singularity in the following form

$$(1.2) \quad b(\cos \theta) \sim \theta^{-2} (\log \theta^{-1})^{\frac{2}{s}-1}, \quad \text{when } \theta \rightarrow 0^+, \quad \text{with } s > 0.$$

This explicit formula was first appeared in the Appendix in [22]. In some sense, the Debye-Yukawa type potentials is a model between the Coulomb potential corresponding to  $s =$

0 and the inverse-power potential: This behavior can be computed from the equations (conservation of energy and angular momentum respectively)

$$\begin{aligned}\frac{1}{2}(\dot{\rho}^2 + \rho^2 \dot{\varphi}^2) + U(\rho) &= \frac{1}{2}V^2 + U(\sigma), \\ \rho^2 \dot{\varphi} &= p(V, \tilde{\theta})V^2\end{aligned}$$

where  $\rho$  and  $\varphi$  are the radial and angular coordinates in the plane of motion and  $p(V, \tilde{\theta})$  is the impact parameter which defines the collision cross section

$$B(|z|, \tilde{\theta}) = |z| \frac{p}{2 \sin \tilde{\theta}} \frac{\partial p}{\partial \tilde{\theta}}$$

and  $z = v - v_*$  is the relative velocity,  $\theta = \pi - 2\tilde{\theta}$  is the deviation angle. For further details on the physics background and the derivation of the Boltzmann equation, we refer to the references [4], [29].

We linearize the Boltzmann equation near the absolute Maxwellian distribution

$$\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}.$$

Let  $f(t, v) = \mu(v) + \sqrt{\mu}(v)g(t, v)$ . Plugging this expression into (1.1), we have

$$\frac{\partial g}{\partial t} + \mathcal{L}[g] = \Gamma(g, g)$$

with

$$\Gamma(g, h) = \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu}g, \sqrt{\mu}h), \quad \mathcal{L}(g) = -\frac{1}{\sqrt{\mu}} [Q(\sqrt{\mu}g, \mu) + Q(\mu, \sqrt{\mu}g)].$$

Then the Cauchy problem (1.1) can be rewritten in the form

$$(1.3) \quad \begin{cases} \partial_t g + \mathcal{L}(g) = \Gamma(g, g), \\ g|_{t=0} = g_0. \end{cases}$$

The linear operator  $\mathcal{L}$  is nonnegative ([14, 15, 16]), with the null space

$$\mathcal{N} = \text{span} \{ \sqrt{\mu}, \sqrt{\mu}v_1, \sqrt{\mu}v_2, \sqrt{\mu}v_3, \sqrt{\mu}|v|^2 \}.$$

Denote by  $\mathbf{P}$  the orthoprojection from  $L^2(\mathbb{R}^3)$  into  $\mathcal{N}$ . Then

$$(\mathcal{L}g, g) = 0 \Leftrightarrow g = \mathbf{P}g.$$

In the case of the inverse-power potential with  $r > 1$ , the regularity of the Boltzmann equation has been studied by numerous papers. Regarding the Cauchy problem (1.1), it is well known that the non-cutoff spatially homogeneous Boltzmann equation enjoys an  $\mathcal{S}(\mathbb{R}^3)$ -regularizing effect for the weak solutions to the Cauchy problem (1.1) (see [6, 22]). For the Gevrey regularity, Ukai showed in [28] that the Cauchy problem for the Boltzmann equation has a unique local solution in Gevrey classes. Then, Desvillettes, Furioli and Terraneo proved in [5] the propagation of Gevrey regularity for solutions of the Boltzmann equation with Maxwellian molecules. For mild singularities, Morimoto and Ukai proved in [21] the Gevrey regularity of smooth Maxwellian decay solutions to the Cauchy problem of the spatially homogeneous Boltzmann equation with a modified kinetic factor. See also [31] for the non-modified case. On the other hand, Lekrine and Xu proved in [13] the property of Gevrey smoothing effect for the weak solutions to the Cauchy problem associated to the radially symmetric spatially homogeneous Boltzmann equation with Maxwellian molecules for  $r > 2$ . This result was then completed by Glangetas and Najeme who established in [8] the analytic smoothing effect in the case when  $1 < r < 2$ . In [1], it has led to the hope that the homogenous Boltzmann equation enjoys similar regularity properties as the heat

equation with a fractional Laplacian. Regarding the linearized Cauchy problem (1.3), it has been proved that the solutions for linearized non-cutoff Boltzmann equation belongs to the symmetric Gelfand-Shilov spaces  $S_{r/2}^{r/2}(\mathbb{R}^3)$  for any positive time, see [14], [17]. The Gelfand-Shilov space  $S_\nu^\nu(\mathbb{R}^3)$  with  $\nu \geq \frac{1}{2}$  can be identify with

$$S_\nu^\nu(\mathbb{R}^3) = \left\{ f \in C^\infty(\mathbb{R}^3); \exists \tau > 0, \|e^{\tau \mathcal{H}^{\frac{1}{2\nu}}} f\|_{L^2} < +\infty \right\}.$$

where  $\mathcal{H}$  is the harmonic oscillator

$$\mathcal{H} = -\Delta + \frac{|v|^2}{4}.$$

For the Cauchy problem (1.3), it has been proved in [16] and [10] that the Cauchy problem for the non-cutoff spatially homogeneous Boltzmann equation with the small initial datum  $g_0 \in L^2(\mathbb{R}^3)$  has a global solution, which belongs to the Gelfand-Shilov class  $S_{r/2}^{r/2}(\mathbb{R}^3)$ .

In the present work, we consider the collision kernel in the Maxwellian molecules case and the angular function  $b$  satisfying the Debye-Yukawa potential (1.2) for some  $s > 0$ . For convenience, we denote

$$(1.4) \quad \beta(\theta) = 2\pi b(\cos \theta) \sin \theta.$$

We study the smoothing effect for the Cauchy problem (1.3) associated to the non-cutoff spatially homogeneous Boltzmann equation with Debye-Yukawa potential (1.2). The singularity of the collision kernel  $b$  endows the linearized Boltzmann operator  $\mathcal{L}$  with the logarithmic regularity property, see Proposition 2.1 in [9], the linearized Debye-Yukawa potential Boltzmann operator  $\mathcal{L}$  was shown to behave as a fractional logarithmic harmonic oscillator  $(\log(\mathcal{H} + 1))^{\frac{s}{2}}$ . The logarithmic regularity theory was first introduced in [18] on the hypoellipticity of the infinitely degenerate elliptic operator and was developed in [23],[24] on the logarithmic Sobolev estimates. Recently, for  $0 < s < 2$ , in [22] it has been shown that weak solutions to the Cauchy problem (1.1) with Debye-Yukawa type interactions enjoy an  $H^\infty$  smoothing property, i.e. starting with arbitrary initial datum  $f_0 \geq 0$ ,

$$\int_{\mathbb{R}^3} f_0(v)(1 + |v|^2 + \log(1 + f_0(v)))dv < +\infty,$$

one has  $f(t, \cdot) \in H^\infty(\mathbb{R}^3)$  for any positive time  $t > 0$ . This result was extended by J.-M. Barbaroux, D. Hundertmark, T. Ried, S. Vugalter in [2]. They showed a stronger regularisation property : for any  $0 < s < 2$ , and for any  $T_0 > 0$ , there exist  $\beta, M > 0$  such that

$$e^{\beta t (\log \langle D_v \rangle)^{\frac{s}{2}}} f(t, \cdot) \in L^2(\mathbb{R}^d)$$

and

$$\sup_{\lim_{\eta \in \mathbb{R}^d}} e^{\beta t (\log \langle D_v \rangle)^{\frac{s}{2}}} |\hat{f}(t, \eta)| \leq M$$

for all  $t \in (0, T_0]$  with  $\langle v \rangle = (1 + |v|^2)^{1/2}$ .

In this paper, we improve the regularisation property (for small initial data). Based upon our recent results [16] and [10] of the Gelfand-Shilov smoothing effect for the homogeneous Boltzmann equation with Maxwellian molecules in the case of the inverse-power potential and the result of [9] for the linear homogeneous Boltzmann equation with Debye-Yukawa potential, we show that, for small initial data, the Cauchy problem (1.3) for the radially symmetric homogeneous Boltzmann equation enjoys the same smoothing effect as the Cauchy problem defined by the evolution equation associated to a fractional logarithmic harmonic oscillator. In order to precise the regularizing effect of the solution for the

Cauchy problem (1.3), we introduce the Shubin spaces. Let  $\tau \in \mathbb{R}$ , we denote by  $Q^\tau(\mathbb{R}^3)$  the spaces introduced by Shubin [27], Ch. IV, 25.3, with norm

$$\|u\|_{Q^\tau(\mathbb{R}^3)} = \left\| \left( -\Delta + \frac{|v|^2}{4} + 1 \right)^{\frac{\tau}{2}} u \right\|_{L^2(\mathbb{R}^3)} = \left\| (\mathcal{H} + 1)^{\frac{\tau}{2}} u \right\|_{L^2(\mathbb{R}^3)}.$$

Now we begin to present our results.

**Theorem 1.1.** *Assume that the Maxwellian collision cross-section  $b(\cdot)$  is given in (1.2) with  $0 < s \leq 2$ , then there exists  $\varepsilon_0 > 0$  such that for any initial datum  $g_0 \in L^2(\mathbb{R}^3) \cap \mathcal{N}^\perp$  with  $\|g_0\|_{L^2(\mathbb{R}^3)}^2 \leq \varepsilon_0$ , the Cauchy problem (1.3) admits a solution which belongs to any Shubin spaces for any  $t > 0$ . Furthermore, there exist  $c_0 > 0$ ,  $C > 0$  such that, for any  $t \geq 0$ ,*

$$(1.5) \quad \|e^{tc_0(\log(\mathcal{H}+1))\frac{\tau}{2}} g\|_{L^2} \leq C e^{-\frac{\lambda_{2,0}t}{4}} \|g_0\|_{L^2(\mathbb{R}^3)},$$

where

$$\lambda_{2,0} = \int_{|\theta| \leq \pi/4} \beta(\theta) (1 - \sin^4 \theta - \cos^4 \theta) d\theta > 0.$$

To be more specific,

1) in the case  $0 < s \leq 2$ :

$$(1.6) \quad \forall t > 0, \quad \|g(t)\|_{Q^{c_0 t}} \leq e^{-\frac{\lambda_{2,0}t}{4}} \|g_0\|_{L^2(\mathbb{R}^3)}.$$

2) in the case  $0 < s < 2$ , there exists a constant  $c_s > 0$  such that for any  $t > 0$ ,

$$(1.7) \quad \forall k \geq 0, \quad \|g(t)\|_{Q^k} \leq e^{-\frac{\lambda_{2,0}t}{4}} e^{c_s(1/t)^{\frac{s}{2-s}} k^{\frac{2}{2-s}}} \|g_0\|_{L^2(\mathbb{R}^3)}.$$

**Remark 1.2.** *We have proved that, if the initial data  $g_0$  is small enough and contained in  $L^2(\mathbb{R}^3)$  in Cauchy problem (1.3), then the global solution for the Cauchy problem (1.1) return to the equilibrium with respect to Shubin space norm.*

**Remark 1.3.** *We think that the regularity properties are optimal, since they are optimal concerning the linearised Cauchy problem (see [9]).*

The rest of the paper is arranged as follows. In Section 2, we introduce the spectral analysis of the linear Boltzmann operator and in Section 3, we establish an upper bounded estimates of the nonlinear operators with an exponential weighted norm. The proof of the main Theorem 1.1 will be presented in Section 4. In Section 5, we provide the proof of the technical Lemma 3.2. In the Appendix 6, we present some identity properties of the Shubin spaces used in this paper and the proof of some technical Lemmas.

## 2. THE SPECTRAL ANALYSIS OF THE BOLTZMANN OPERATORS

**2.1. Diagonalization of linear operators.** We first recall the spectral decomposition of linear Boltzmann operator. In the cutoff case, that is, when  $b(\cos \theta) \sin \theta \in L^1([0, \frac{\pi}{2}])$ , it was shown in [30] that

$$\mathcal{L}(\varphi_{n,l,m}) = \lambda_{n,l} \varphi_{n,l,m}, \quad n, l \in \mathbb{N}, \quad m \in \mathbb{Z}, \quad |m| \leq l.$$

This diagonalization of the linearized Boltzmann operator with Maxwellian molecules holds as well in the non-cutoff case, (see [3, 4, 7, 14, 15]). Where

$$\lambda_{n,l} = \int_{|\theta| \leq \frac{\pi}{4}} \beta(\theta) \left( 1 + \delta_{n,0} \delta_{l,0} - (\sin \theta)^{2n+l} P_l(\sin \theta) - (\cos \theta)^{2n+l} P_l(\cos \theta) \right) d\theta,$$

the eigenfunctions are

$$\varphi_{n,l,m}(v) = \left( \frac{n!}{\sqrt{2}\Gamma(n+l+3/2)} \right)^{1/2} \left( \frac{|v|}{\sqrt{2}} \right)^l e^{-\frac{|v|^2}{4}} L_n^{(l+1/2)} \left( \frac{|v|^2}{2} \right) Y_l^m \left( \frac{v}{|v|} \right),$$

where  $\Gamma(\cdot)$  is the standard Gamma function, for any  $x > 0$ ,

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-x} dx.$$

The  $l^{\text{th}}$ -Legendre polynomial  $P_l$  and the Laguerre polynomial  $L_n^{(\alpha)}$  of order  $\alpha$ , degree  $n$  (see [26]) read,

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l, \text{ where } |x| \leq 1;$$

$$L_n^{(\alpha)}(x) = \sum_{r=0}^n (-1)^{n-r} \frac{\Gamma(\alpha + n + 1)}{r!(n-r)!\Gamma(\alpha + n - r + 1)} x^{n-r}.$$

For any unit vector  $\sigma = (\cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi)$  with  $\theta \in [0, \pi]$  and  $\phi \in [0, 2\pi]$ , the orthonormal basis of spherical harmonics  $Y_l^m(\sigma)$  is

$$Y_l^m(\sigma) = N_{l,m} P_l^{|m|}(\cos \theta) e^{im\phi}, \quad |m| \leq l,$$

where the normalisation factor is given by

$$N_{l,m} = \sqrt{\frac{2l+1}{4\pi} \cdot \frac{(l-|m|)!}{(l+|m|)!}}$$

and  $P_l^{|m|}$  is the associated Legendre functions of the first kind of order  $l$  and degree  $|m|$  with

$$(2.1) \quad P_l^{|m|}(x) = (1-x^2)^{\frac{|m|}{2}} \left( \frac{d}{dx} \right)^{|m|} P_l(x).$$

The family  $(Y_l^m(\sigma))_{l \geq 0, |m| \leq l}$  constitutes an orthonormal basis of the space  $L^2(\mathbb{S}^2, d\sigma)$  with  $d\sigma$  being the surface measure on  $\mathbb{S}^2$  (see [12], [25]). Noting that  $\{\varphi_{n,l,m}\}$  consist an orthonormal basis of  $L^2(\mathbb{R}^3)$  composed of eigenvectors of the harmonic oscillator (see[3], [15])

$$\mathcal{H}(\varphi_{n,l,m}) = (2n + l + \frac{3}{2}) \varphi_{n,l,m}.$$

As a special case,  $\{\varphi_{n,0,0}\}$  consist an orthonormal basis of  $L_{rad}^2(\mathbb{R}^3)$  in the radially symmetric function space (see [16]) and

$$\mathcal{H}(\varphi_{n,0,0}) = (2n + \frac{3}{2}) \varphi_{n,0,0}.$$

We have that, for suitable functions  $g$ ,

$$\mathcal{L}(g) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^l \lambda_{n,l} g_{n,l,m} \varphi_{n,l,m},$$

where  $g_{n,l,m} = (g, \varphi_{n,l,m})_{L^2(\mathbb{R}^3)}$ , and

$$\mathcal{H}(g) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^l (2n + l + \frac{3}{2}) g_{n,l,m} \varphi_{n,l,m}.$$

Using this spectral decomposition, the definition of  $(\log(\mathcal{H} + 1))^{\frac{2}{s}}$ ,  $e^{c(\log(\mathcal{H} + 1))^{\frac{2}{s}}}$ ,  $e^{c\mathcal{L}}$  is then classical.

**2.2. Triangular effect of the non linear operators.** We study now the algebra property of the nonlinear terms

$$\Gamma(\varphi_{n,0,0}, \varphi_{m,0,0}),$$

By the same proof of Proposition 2.1 in [10], we have the following triangular effect for the nonlinear Boltzmann operators on the basis  $\{\varphi_{n,0,0}\}$ .

**Proposition 2.1.** *The following algebraic identities hold,*

$$\begin{aligned} (i_1) \quad \Gamma(\varphi_{0,0,0}, \varphi_{m,0,0}) &= \left( \int_0^{\frac{\pi}{4}} \beta(\theta) ((\cos \theta)^{2m} - 1) d\theta \right) \varphi_{m,0,0}, \quad m \in \mathbb{N}; \\ (i_2) \quad \Gamma(\varphi_{n,0,0}, \varphi_{0,0,0}) &= \left( \int_0^{\frac{\pi}{4}} \beta(\theta) ((\sin \theta)^{2n} - \delta_{0,n}) d\theta \right) \varphi_{n,0,0}, \quad n \in \mathbb{N}; \\ (ii) \quad \Gamma(\varphi_{n,0,0}, \varphi_{m,0,0}) &= \mu_{n,m} \varphi_{n+m,0,0}, \quad \text{for } n \geq 1, m \geq 1, \end{aligned}$$

where

$$(2.2) \quad \mu_{n,m} = \sqrt{\frac{(2n+2m+1)!}{(2n+1)!(2m+1)!}} \left( \int_0^{\frac{\pi}{4}} \beta(\theta) (\sin \theta)^{2n} (\cos \theta)^{2m} d\theta \right).$$

**Remark 2.2.** Obviously, we can deduce from  $(i_1)$  and  $(i_2)$  of Proposition 2.1 that

$$\forall n \in \mathbb{N}, \quad \Gamma(\varphi_{0,0,0}, \varphi_{n,0,0}) + \Gamma(\varphi_{n,0,0}, \varphi_{0,0,0}) = -\lambda_{n,0} \varphi_{n,0,0}.$$

Where  $\lambda_{0,0} = \lambda_{1,0} = 0$  and for  $n \geq 2$ ,

$$\lambda_{n,0} = \int_0^{\frac{\pi}{4}} \beta(\theta) (1 - (\cos \theta)^{2n} - (\sin \theta)^{2n}) d\theta.$$

From Proposition 2.1 in [9], there exists a  $c_0 > 0$  dependent only on  $s$ , such that, for  $n \geq 2$ ,

$$(2.3) \quad c_0 (\log(2n + \frac{5}{2}))^{\frac{2}{s}} \leq \lambda_{n,0} \leq \frac{1}{c_0} (\log(2n + \frac{5}{2}))^{\frac{2}{s}}.$$

This shows that the linearized radially symmetric spatially homogeneous Boltzmann operator with Debye-Yukawa potential was shown to behave as a fractional logarithmic harmonic oscillator  $(\log(\mathcal{H} + 1))^{\frac{2}{s}}$ .

**2.3. Explicit solution of the Cauchy problem.** Now we solve explicitly the Cauchy problem associated to the non-cutoff radial symmetric spatially homogeneous Boltzmann equation with Maxwellian molecules for a small  $L^2$ -initial radial data.

We search a radial solution to the Cauchy problem (1.3) in the form

$$g(t) = \sum_{n=0}^{+\infty} g_n(t) \varphi_{n,0,0},$$

with initial data

$$g(0) = \sum_{n=0}^{+\infty} (g_0, \varphi_{n,0,0})_{L^2(\mathbb{R}^3)} \varphi_{n,0,0} \in L^2(\mathbb{R}^3),$$

where

$$g_n(t) = (g(t), \varphi_{n,0,0})_{L^2(\mathbb{R}^3)}.$$

It follows from Proposition 2.1 and Remark 2.2 that, for convenable radial symmetric function  $g$ , we have

$$\begin{aligned}\Gamma(g, g) &= - \sum_{n=0}^{+\infty} g_0(t) g_n(t) \lambda_{n,0} \varphi_{n,0,0} \\ &\quad + \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} g_n(t) g_m(t) \mu_{n,m} \varphi_{m+n,0,0},\end{aligned}$$

where  $\mu_{n,m}$  was defined in (2.2). This implies that,

$$\Gamma(g, g) = \sum_{n=0}^{+\infty} \left[ -g_0(t) g_n(t) \lambda_{n,0} + \sum_{\substack{k+l=n \\ k \geq 1, l \geq 1}} g_k(t) g_l(t) \mu_{k,l} \right] \varphi_{n,0,0}.$$

For radial symmetric function  $g$ , we also have

$$\mathcal{L}g = \sum_{n=0}^{+\infty} \lambda_{n,0} g_n(t) \varphi_{n,0,0}.$$

Formally, we take inner product with  $\varphi_{n,0,0}$  on both sides of (1.3), we find that the functions  $\{g_n(t)\}$  satisfy the following infinite system of the differential equations

$$(2.4) \quad \partial_t g_n(t) + \lambda_{n,0} g_n(t) = -g_0(t) g_n(t) \lambda_{n,0} + \sum_{\substack{k+l=n \\ k \geq 1, l \geq 1}} g_k(t) g_l(t) \mu_{k,l}, \quad \forall n \in \mathbb{N},$$

with initial data

$$g_n(0) = (g_0, \varphi_{n,0,0})_{L^2(\mathbb{R}^3)}.$$

Consider that  $g_0 \in \mathcal{N}^\perp$ , we have

$$g_0(0) = g_1(0) = 0.$$

The infinite system of the differential equations (2.4) reduces to be

$$(2.5) \quad \begin{cases} g_0(t) = g_1(t) = 0, \\ \partial_t g_n(t) + \lambda_{n,0} g_n(t) = \sum_{\substack{k+l=n \\ k \geq 1, l \geq 1}} g_k(t) g_l(t) \mu_{k,l}, \quad \forall n \geq 2, \\ g_n(0) = (g_0, \varphi_{n,0,0})_{L^2(\mathbb{R}^3)}. \end{cases}$$

On the right hand side of the second equation in (2.5), the indices  $k$  and  $l$  are always less than  $n$ , then this system of the differential equations is triangular, which can be explicitly solved while solving a sequence of linear differential equations.

The proof of Theorem 1.1 is reduced to prove the convergence of following series

$$(2.6) \quad g(t) = \sum_{n=2}^{+\infty} g_n(t) \varphi_{n,0,0}$$

in the convenable function space.



### 3. THE SHARP TRILINEAR ESTIMATES FOR THE RADially SYMMETRIC BOLTZMANN OPERATOR

To prove the convergence of the formal solution obtained in the precedent section, we need to estimate the following trilinear terms

$$(\Gamma(f, g), h)_{L^2(\mathbb{R}^3)}, \quad f, g, h \in \mathcal{S}_r(\mathbb{R}^3) \cap \mathcal{N}^\perp.$$

By a proof similar to that in Lemma 3.1 in [16], we present the properties of the eigenvalues of the linearized radially symmetric Boltzmann operator  $\mathcal{L}$ , which is a basic tool in the proof of the trilinear estimate with exponential weighted.

**Lemma 3.1.** *The eigenvalues of the linearized radially symmetric Boltzmann operator  $\mathcal{L}$*

$$\lambda_{n,0} = \int_{|\theta| \leq \frac{\pi}{4}} \beta(\theta) (1 - (\sin \theta)^{2n} - (\cos \theta)^{2n}) d\theta$$

satisfy to the following estimate

$$\forall k, l \geq 2, \quad \lambda_{k,0} + \lambda_{l,0} > \lambda_{k+l,0}.$$

*Proof.* Since  $\beta(\theta) > 0$ , we only need to prove that, for  $\theta \in [-\frac{\pi}{4}, \frac{\pi}{4}] \setminus 0$ ,  $\forall k, l \geq 2$ ,

$$(3.1) \quad 1 + (\cos \theta)^{2k+2l} + (\sin \theta)^{2k+2l} - (\cos \theta)^{2k} - (\sin \theta)^{2k} - (\cos \theta)^{2l} - (\sin \theta)^{2l} > 0.$$

By a proof similar to that in Lemma 3.1 in [16], the estimate (3.1) follows. This ends the proof of Lemma 3.1.  $\square$

The following lemma is instrumental in the proof of the trilinear estimates:

**Lemma 3.2.** *For  $n \geq 2$  and  $\mu_{k,l}$  was defined in (2.2) with  $k, l \in \mathbb{N}$ ,  $0 < s \leq 2$ , we have*

$$(3.2) \quad \sum_{\substack{k+l=n \\ k \geq 1, l \geq 1}} \frac{|\mu_{k,l}|^2}{(\log(2l + \frac{5}{2}))^{2/s}} \lesssim (\log(2n + \frac{5}{2}))^{2/s}.$$

We prove this Lemma in Section 5.

The sharp trilinear estimates for the radially symmetric Boltzmann operator can be derived from the result of Lemma 3.2.

**Proposition 3.3.** *For  $0 < s \leq 2$ , there exists a positive  $C > 0$ , such that for all  $f, g, h \in \mathcal{S}_r(\mathbb{R}^3) \cap \mathcal{N}^\perp$ ,*

$$|(\Gamma(f, g), h)_{L^2}| \leq C \|f\|_{L^2} \|(\log(\mathcal{H} + 1))^{\frac{1}{s}} g\|_{L^2} \|(\log(\mathcal{H} + 1))^{\frac{1}{s}} h\|_{L^2},$$

and for any  $t \geq 0$ ,  $n \geq 2$ ,

$$\begin{aligned} & |(\Gamma(f, g), e^{t\mathcal{L}} \mathbb{S}_n h)_{L^2}| \\ & \leq C \|e^{\frac{t}{2}\mathcal{L}} \mathbb{S}_{n-2} f\|_{L^2} \|e^{\frac{t}{2}\mathcal{L}} (\log(\mathcal{H} + 1))^{\frac{1}{s}} g \mathbb{S}_{n-2} g\|_{L^2} \|e^{\frac{t}{2}\mathcal{L}} (\log(\mathcal{H} + 1))^{\frac{1}{s}} \mathbb{S}_n h\|_{L^2}, \end{aligned}$$

where  $\mathcal{L}$  is the linearized non-cutoff Boltzmann operator,  $\mathcal{H} = -\Delta + \frac{|\mathbf{v}|^2}{4}$  is the 3-dimensional harmonic oscillator and  $\mathbb{S}_n$  is the orthogonal projector onto the  $n+1$  energy levels

$$\mathbb{S}_n f = \sum_{k=0}^n (f, \varphi_{k,0,0})_{L^2} \varphi_{k,0,0}, \quad e^{t\mathcal{L}} \mathbb{S}_n f = \sum_{k=0}^n e^{\lambda_{k,0} t} (f, \varphi_{k,0,0})_{L^2} \varphi_{k,0,0}.$$

*Proof.* Let  $f, g, h \in \mathcal{S}_r(\mathbb{R}^3) \cap \mathcal{N}^\perp$  be the radial Schwartz functions, by using the spectral decomposition, we obtain

$$f = \sum_{n=2}^{+\infty} (f, \varphi_{n,0,0})_{L^2} \varphi_{n,0,0}, \quad g = \sum_{n=2}^{+\infty} (g, \varphi_{n,0,0})_{L^2} \varphi_{n,0,0}, \quad h = \sum_{n=2}^{+\infty} (h, \varphi_{n,0,0})_{L^2} \varphi_{n,0,0}.$$

In convenience, we rewrite  $f_n = (f, \varphi_{n,0,0})_{L^2}$ ,  $g_n = (g, \varphi_{n,0,0})_{L^2}$ ,  $h_n = (h, \varphi_{n,0,0})_{L^2}$ . We can deduce from Proposition 2.1 that,

$$\begin{aligned}\Gamma(f, g) &= \sum_{k=2}^{+\infty} \sum_{l=2}^{+\infty} f_k g_l \Gamma(\varphi_{k,0,0}, \varphi_{l,0,0}) \\ &= \sum_{k=2}^{+\infty} \sum_{l=2}^{+\infty} f_k g_l \mu_{k,l} \varphi_{k+l,0,0} \\ &= \sum_{n=4}^{+\infty} \left( \sum_{\substack{k+l=n \\ k \geq 2, l \geq 2}} \mu_{k,l} f_k g_l \right) \varphi_{n,0,0}.\end{aligned}$$

Applying the orthogonal property of  $\varphi_{n,0,0}$ , it follows that,

$$(\Gamma(f, g), h)_{L^2} = \sum_{n=4}^{+\infty} \left( \sum_{\substack{k+l=n \\ k \geq 2, l \geq 2}} \mu_{k,l} f_k g_l \right) h_n.$$

We use the Cauchy-Schwarz inequality,

$$\begin{aligned}& |(\Gamma(f, g), h)_{L^2}| \\ & \leq \sum_{k=2}^{+\infty} \sum_{l=2}^{+\infty} |\mu_{k,l}| |f_k| |g_l| |h_{k+l}| \\ & \leq \left( \sum_{l=2}^{+\infty} \left( \log(2l + \frac{5}{2}) \right)^{\frac{2}{s}} |g_l|^2 \right)^{\frac{1}{2}} \left( \sum_{l=2}^{+\infty} \frac{1}{\left( \log(2l + \frac{5}{2}) \right)^{\frac{2}{s}}} \left( \sum_{k=2}^{+\infty} |\mu_{k,l}| |f_k| |h_{k+l}| \right)^2 \right)^{\frac{1}{2}} \\ & \leq \left( \sum_{l=2}^{+\infty} \left( \log(2l + \frac{5}{2}) \right)^{\frac{2}{s}} |g_l|^2 \right)^{\frac{1}{2}} \left( \sum_{k=2}^{+\infty} |f_k|^2 \right)^{\frac{1}{2}} \\ & \quad \times \left( \sum_{l=2}^{+\infty} \sum_{k=2}^{+\infty} \frac{|\mu_{k,l}|^2}{\left( \log(2l + \frac{5}{2}) \right)^{\frac{2}{s}}} |h_{k+l}|^2 \right)^{\frac{1}{2}} \\ & = \|(\log(\mathcal{H} + 1))^{\frac{1}{s}} g\|_{L^2} \|f\|_{L^2} \left( \sum_{n=4}^{+\infty} \left( \sum_{\substack{k+l=n \\ k \geq 2, l \geq 2}} \frac{|\mu_{k,l}|^2}{\left( \log(2l + \frac{5}{2}) \right)^{\frac{2}{s}}} \right) |h_n|^2 \right)^{\frac{1}{2}}.\end{aligned}$$

It follows from Lemma 3.2 that

$$\sum_{\substack{k+l=n \\ k \geq 2, l \geq 2}} \frac{|\mu_{k,l}|^2}{\left( \log(2l + \frac{5}{2}) \right)^{\frac{2}{s}}} \lesssim \left( \log(2n + \frac{5}{2}) \right)^{\frac{2}{s}},$$

then

$$\left( \sum_{n=4}^{+\infty} \left( \sum_{\substack{k+l=n \\ k \geq 2, l \geq 2}} \frac{|\mu_{k,l}|^2}{\left( \log(2l + \frac{5}{2}) \right)^{\frac{2}{s}}} \right) |h_n|^2 \right)^{\frac{1}{2}} \lesssim \|(\log(\mathcal{H} + 1))^{\frac{1}{s}} h\|_{L^2}.$$

This implies that, there exists  $C > 0$ ,

$$|(\Gamma(f, g), h)_{L^2}| \leq C \|(\log(\mathcal{H} + 1))^{\frac{1}{s}} g\|_{L^2} \|f\|_{L^2} \|(\log(\mathcal{H} + 1))^{\frac{1}{s}} h\|_{L^2}.$$

On the other hand, we consider the inequality with exponential weighted and apply the orthogonal property of  $\varphi_{n,0,0}$  again that

$$(\Gamma(f, g), e^{t\mathcal{L}}\mathbb{S}_n h) = \sum_{m=4}^n e^{\lambda_{m,0}t} \left( \sum_{\substack{k+l=m \\ k \geq 2, l \geq 2}} \mu_{k,l} f_k g_l \right) h_m.$$

Then

$$\begin{aligned} & |(\Gamma(f, g), e^{t\mathcal{L}}\mathbb{S}_n h)| \\ & \leq \sum_{l=2}^{n-2} \sum_{k=2}^{n-l} |g_l| |\mu_{k,l}| |f_k| |h_{k+l}| e^{\lambda_{k+l,0}t} \\ & \leq \left( \sum_{l=2}^{n-2} e^{\lambda_{l,0}t} \left( \log(2l + \frac{5}{2}) \right)^{\frac{2}{s}} |g_l|^2 \right)^{\frac{1}{2}} \\ & \quad \times \left( \sum_{l=2}^{n-2} \frac{e^{-\lambda_{l,0}t}}{\left( \log(2l + \frac{5}{2}) \right)^{\frac{2}{s}}} \left( \sum_{k=2}^{n-l} e^{\lambda_{k+l,0}t} |\mu_{k,l}| |f_k| |h_{k+l}| \right)^2 \right)^{\frac{1}{2}} \\ & \leq \left( \sum_{l=2}^{n-2} e^{\lambda_{l,0}t} \left( \log(2l + \frac{5}{2}) \right)^{\frac{2}{s}} |g_l|^2 \right)^{\frac{1}{2}} \left( \sum_{k=2}^{n-2} e^{\lambda_{k,0}t} |f_k|^2 \right)^{\frac{1}{2}} \\ & \quad \times \left( \sum_{l=2}^{n-2} \frac{1}{\left( \log(2l + \frac{5}{2}) \right)^{\frac{2}{s}}} \sum_{k=2}^{n-l} e^{2\lambda_{k+l,0}t - \lambda_{l,0}t - \lambda_{k,0}t} |\mu_{k,l}|^2 |h_{k+l}|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Since by Lemma 3.1, for all  $k \geq 2, l \geq 2$ ,

$$\lambda_{k+l,0} - \lambda_{l,0} - \lambda_{k,0} \leq 0,$$

one can verify that, for  $t \geq 0$ ,

$$\begin{aligned} & |(\Gamma(f, g), e^{t\mathcal{L}}\mathbb{S}_n h)| \\ & \leq \|e^{\frac{t}{2}\mathcal{L}}\mathbb{S}_{n-2}f\|_{L^2} \|e^{\frac{t}{2}\mathcal{L}}(\log(\mathcal{H} + 1))^{\frac{1}{s}}\mathbb{S}_{n-2}g\|_{L^2} \\ & \quad \times \left( \sum_{l=2}^{n-2} \frac{1}{\left( \log(2l + \frac{5}{2}) \right)^{\frac{2}{s}}} \sum_{k=2}^{n-l} e^{\lambda_{k+l,0}t} |\mu_{k,l}|^2 |h_{k+l}|^2 \right)^{\frac{1}{2}} \\ & = \|e^{\frac{t}{2}\mathcal{L}}\mathbb{S}_{n-2}f\|_{L^2} \|e^{\frac{t}{2}\mathcal{L}}(\log(\mathcal{H} + 1))^{\frac{1}{s}}\mathbb{S}_{n-2}g\|_{L^2} \\ & \quad \times \left( \sum_{m=4}^n e^{\lambda_{m,0}t} \left( \sum_{\substack{k+l=m \\ k \geq 2, l \geq 2}} \frac{|\mu_{k,l}|^2}{\left( \log(2l + \frac{5}{2}) \right)^{\frac{2}{s}}} \right) |h_m|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

By using Lemma 3.2 again that, for  $m \geq 4$ ,

$$\sum_{\substack{k+l=m \\ k \geq 2, l \geq 2}} \frac{|\mu_{k,l}|^2}{\left( \log(2l + \frac{5}{2}) \right)^{\frac{2}{s}}} \lesssim \left( \log(2m + \frac{5}{2}) \right)^{\frac{2}{s}}.$$

We conclude, for  $t \geq 0$ ,  $n \geq 2$ ,

$$\begin{aligned} & |(\Gamma(f, g), e^{t\mathcal{L}}\mathbb{S}_n h)_{L^2}| \\ & \leq C \|e^{\frac{t}{2}\mathcal{L}}\mathbb{S}_{n-2}f\|_{L^2} \|e^{\frac{t}{2}\mathcal{L}}(\log(\mathcal{H}+1))^{\frac{1}{2}}\mathbb{S}_{n-2}g\|_{L^2} \|e^{\frac{t}{2}\mathcal{L}}(\log(\mathcal{H}+1))^{\frac{1}{2}}\mathbb{S}_n h\|_{L^2}. \end{aligned}$$

This ends the proof of Proposition 3.3.  $\square$

**Remark 3.4.** From the remark 3.2, the linearized radially symmetric spatially homogeneous Boltzmann operator with Debye-Yukawa potential  $\mathcal{L}$  was shown to behave as a fractional logarithmic harmonic oscillator  $(\log(\mathcal{H}+1))^{\frac{1}{2}}$ , one can verify from Proposition 3.3 that there exists a constant  $C_1$  such that

$$|(\Gamma(f, g), e^{t\mathcal{L}}\mathbb{S}_n h)_{L^2}| \leq C_1 \|e^{\frac{t}{2}\mathcal{L}}\mathbb{S}_{n-2}f\|_{L^2} \|e^{\frac{t}{2}\mathcal{L}}\mathcal{L}^{\frac{1}{2}}\mathbb{S}_{n-2}g\|_{L^2} \|e^{\frac{t}{2}\mathcal{L}}\mathcal{L}^{\frac{1}{2}}\mathbb{S}_n h\|_{L^2}.$$

#### 4. THE PROOF OF THE MAIN THEOREM

In this section, we study the convergence of the formal solutions obtained on Section 2 with small  $L^2$  initial data which end the proof of Theorem 1.1.

**4.1. The uniform estimate.** Let  $\{g_n(t)\}$  be the solution of (2.6), for any  $2 \leq N \in \mathbb{N}$ , set

$$(4.1) \quad \mathbb{S}_N g(t) = \sum_{n=2}^N g_n(t) \varphi_{n,0,0},$$

then  $\mathbb{S}_N g(t), e^{t\mathcal{L}}\mathbb{S}_N g(t) \in \mathcal{S}_r(\mathbb{R}^3) \cap \mathcal{N}^\perp$ ,

Multiplying  $e^{\lambda_{n,0,t}} \overline{g_n}(t)$  on both sides of (2.4) and take summation for  $2 \leq n \leq N$ , then Proposition 2.1 and the orthogonality of the basis  $\{\varphi_{n,0,0}\}_{n \in \mathbb{N}}$  imply that

$$\begin{aligned} & \left( \partial_t (\mathbb{S}_N g)(t), e^{t\mathcal{L}}\mathbb{S}_N g(t) \right)_{L^2(\mathbb{R}^3)} + \left( \mathcal{L}(\mathbb{S}_N g)(t), e^{t\mathcal{L}}\mathbb{S}_N g(t) \right)_{L^2(\mathbb{R}^3)} \\ & = \left( \Gamma((\mathbb{S}_N g), (\mathbb{S}_N g)), e^{t\mathcal{L}}\mathbb{S}_N g(t) \right)_{L^2(\mathbb{R}^3)}. \end{aligned}$$

Since  $\mathbb{S}_N g(t) \in \mathcal{S}_r(\mathbb{R}^3) \cap \mathcal{N}^\perp$ , we have

$$\left( \mathcal{L}(\mathbb{S}_N g)(t), e^{t\mathcal{L}}\mathbb{S}_N g(t) \right)_{L^2(\mathbb{R}^3)} = \|e^{\frac{t}{2}\mathcal{L}}\mathcal{L}^{\frac{1}{2}}\mathbb{S}_N g(t)\|_{L^2(\mathbb{R}^3)}^2,$$

we then obtain that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|e^{\frac{t}{2}\mathcal{L}}\mathbb{S}_N g(t)\|_{L^2}^2 + \frac{1}{2} \|e^{\frac{t}{2}\mathcal{L}}\mathcal{L}^{\frac{1}{2}}\mathbb{S}_N g(t)\|_{L^2(\mathbb{R}^3)}^2 \\ & = \left( \Gamma((\mathbb{S}_N g), (\mathbb{S}_N g)), e^{t\mathcal{L}}\mathbb{S}_N g(t) \right)_{L^2}. \end{aligned}$$

It follows from Remark 3.4 that, for any  $N \geq 2$ ,  $t \geq 0$ ,

$$(4.2) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|e^{\frac{t}{2}\mathcal{L}}\mathbb{S}_N g\|_{L^2}^2 + \frac{1}{2} \|e^{\frac{t}{2}\mathcal{L}}\mathcal{L}^{\frac{1}{2}}\mathbb{S}_N g\|_{L^2}^2 \\ & \leq C_1 \|e^{\frac{t}{2}\mathcal{L}}\mathbb{S}_{N-2}g\|_{L^2} \|e^{\frac{t}{2}\mathcal{L}}\mathcal{L}^{\frac{1}{2}}\mathbb{S}_N g\|_{L^2}^2. \end{aligned}$$

**Proposition 4.1.** There exists  $\epsilon_0 > 0$  such that for all  $0 < \epsilon \leq \epsilon_0$ ,  $g_0 \in L^2 \cap \mathcal{N}^\perp$  with  $\|g_0\|_{L^2} \leq \epsilon$ ,

$$\|e^{\frac{t}{2}\mathcal{L}}\mathbb{S}_N g(t)\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2} \int_0^t \|e^{\frac{\tau}{2}\mathcal{L}}\mathcal{L}^{\frac{1}{2}}\mathbb{S}_N g(\tau)\|_{L^2}^2 d\tau \leq \|g_0\|_{L^2(\mathbb{R}^3)}^2,$$

for any  $t \geq 0$ ,  $N \geq 0$ .

*Proof.* We prove the Proposition by induction on  $N$ .

**1). For  $N \leq 2$ .** we have  $\|e^{\frac{t}{2}\mathcal{L}}\mathbb{S}_0g\|_{L^2}^2 = |g_0(t)|^2 = 0$ ,  $\|e^{\frac{t}{2}\mathcal{L}}\mathbb{S}_1g\|_{L^2}^2 = |g_0(t)|^2 + |g_1(t)|^2 = 0$ , and

$$\|e^{\frac{t}{2}\mathcal{L}}\mathbb{S}_2g\|_{L^2}^2 = e^{\lambda_{0,2}t}|g_2(t)|^2 = e^{-\lambda_{0,2}t}|g_2(0)|^2 \leq |(g_0, \varphi_{2,0,0})_{L^2(\mathbb{R}^3)}|^2 \leq \|g_0\|_{L^2(\mathbb{R}^3)}^2.$$

**2). For  $N > 2$ .** We want to prove that

$$\|e^{\frac{t}{2}\mathcal{L}}\mathbb{S}_{N-1}g\|_{L^2} \leq \epsilon \leq \epsilon_0,$$

imply

$$\|e^{\frac{t}{2}\mathcal{L}}\mathbb{S}_Ng\|_{L^2} \leq \epsilon.$$

Take now  $\epsilon_0 > 0$  such that,

$$0 < \epsilon_0 \leq \frac{1}{4C_1}$$

where  $C_1$  is defined in Remark 3.4. Then we deduce from (4.2) that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|e^{\frac{t}{2}\mathcal{L}}\mathbb{S}_Ng(t)\|_{L^2}^2 + \frac{1}{2} \|e^{\frac{t}{2}\mathcal{L}}\mathcal{L}^{\frac{1}{2}}\mathbb{S}_Ng\|_{L^2}^2 \\ \leq C_1 \|e^{\frac{t}{2}\mathcal{L}}\mathbb{S}_{N-2}g\|_{L^2} \|e^{\frac{t}{2}\mathcal{L}}\mathcal{L}^{\frac{1}{2}}\mathbb{S}_Ng\|_{L^2} \\ \leq \frac{1}{4} \|e^{\frac{t}{2}\mathcal{L}}\mathcal{L}^{\frac{1}{2}}\mathbb{S}_Ng\|_{L^2}^2, \end{aligned}$$

therefore,

$$(4.3) \quad \frac{d}{dt} \|e^{\frac{t}{2}\mathcal{L}}\mathbb{S}_Ng(t)\|_{L^2}^2 + \frac{1}{2} \|e^{\frac{t}{2}\mathcal{L}}\mathcal{L}^{\frac{1}{2}}\mathbb{S}_Ng\|_{L^2}^2 \leq 0.$$

This ends the proof of the Proposition 4.1.  $\square$

**4.2. Existence of the weak solution.** We prove now the convergence of the sequence

$$g(t) = \sum_{n=2}^{+\infty} g_n(t) \varphi_{n,0,0}$$

defined in (2.6). For all  $N \geq 2$ ,  $\mathbb{S}_Ng(t)$  satisfies the following Cauchy problem

$$(4.4) \quad \begin{cases} \partial_t \mathbb{S}_Ng + \mathcal{L}(\mathbb{S}_Ng) = \mathbb{S}_N \Gamma(\mathbb{S}_Ng, \mathbb{S}_Ng), \\ \mathbb{S}_Ng(0) = \sum_{n=2}^N (g_0, \varphi_{n,0,0})_{L^2(\mathbb{R}^3)} \varphi_{n,0,0}. \end{cases}$$

By Proposition 4.1 and the orthogonality of the basis  $(\varphi_{n,0,0})_{n \in \mathbb{N}}$ , for all  $t > 0$ ,

$$\sum_{n=2}^N e^{\lambda_{n,0}t} |g_n(t)|^2 + \frac{1}{2} \int_0^t \left( \sum_{n=2}^N e^{\lambda_{n,0}\tau} \lambda_{n,0} |g(\tau)|^2 \right) d\tau \leq \|g_0\|_{L^2(\mathbb{R}^3)}^2.$$

Since  $\lambda_{n,0} > 0$ , for all  $t \geq 0$ , we have

$$(4.5) \quad \sum_{n=2}^N |g_n(t)|^2 + \frac{1}{2} \int_0^t \left( \sum_{n=2}^N \lambda_{n,0} |g(\tau)|^2 \right) d\tau \leq \|g_0\|_{L^2(\mathbb{R}^3)}^2.$$

The orthogonality of the basis  $(\varphi_{n,0,0})_{n \in \mathbb{N}}$  implies that

$$\|\mathbb{S}_Ng(t)\|_{L^2(\mathbb{R}^3)}^2 = \sum_{n=2}^N |g_n(t)|^2, \quad \|\mathcal{L}^{\frac{1}{2}}\mathbb{S}_Ng(t)\|_{L^2(\mathbb{R}^3)}^2 = \sum_{n=2}^N \lambda_{n,0} |g_n(t)|^2.$$

By using the monotone convergence theorem, the sequence

$$g(t) = \sum_{n=2}^{+\infty} g_n(t) \varphi_{n,0,0}$$

is convergent and for any  $t \geq 0$ ,

$$\lim_{N \rightarrow \infty} \|\mathbb{S}_N g - g\|_{L^\infty([0,t]; L^2(\mathbb{R}^3))} = 0$$

and

$$\lim_{N \rightarrow \infty} \|\mathcal{L}^{\frac{1}{2}}(\mathbb{S}_N g - g)\|_{L^2([0,t]; L^2(\mathbb{R}^3))} = 0.$$

For any  $\phi(t) \in C^1(\mathbb{R}_+, \mathcal{S}(\mathbb{R}^3))$ , we have

$$\begin{aligned} & \left| \int_0^t \left( \mathbb{S}_N \Gamma(\mathbb{S}_N g, \mathbb{S}_N g) - \Gamma(g, g), \phi(\tau) \right)_{L^2(\mathbb{R}^3)} d\tau \right| \\ & \leq \left| \int_0^t \left( \Gamma(\mathbb{S}_N g, \mathbb{S}_N g), \mathbb{S}_N \phi(\tau) - \phi(\tau) \right)_{L^2} d\tau \right| \\ & \quad + \left| \int_0^t \left( \Gamma(\mathbb{S}_N g - g, \mathbb{S}_N g), \phi(\tau) \right)_{L^2} d\tau \right| + \left| \int_0^t \left( \Gamma(g, \mathbb{S}_N g - g), \phi(\tau) \right)_{L^2} d\tau \right|. \end{aligned}$$

By the estimate (4.5) and the orthogonality of the basis  $(\varphi_{n,0,0})_{n \in \mathbb{N}}$ , one can verify that,

$$\begin{aligned} & \left| \int_0^t \left( \mathbb{S}_N \Gamma(\mathbb{S}_N g, \mathbb{S}_N g) - \Gamma(g, g), \phi(\tau) \right)_{L^2(\mathbb{R}^3)} d\tau \right| \\ & \leq C \int_0^t \|\mathbb{S}_N g\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} \mathbb{S}_N g\|_{L^2} \left\| \mathcal{L}^{\frac{1}{2}} (\mathbb{S}_N \phi - \phi) \right\|_{L^2(\mathbb{R}^3)} dt \\ & \quad + C \int_0^t \|\mathbb{S}_N g - g\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} \mathbb{S}_N g\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} \phi\|_{L^2(\mathbb{R}^3)} dt \\ & \quad + C \int_0^t \|g\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} (\mathbb{S}_N g - g)\|_{L^2(\mathbb{R}^3)} \|\mathcal{L}^{\frac{1}{2}} \phi\|_{L^2(\mathbb{R}^3)} dt \\ & \leq C \|g_0\|_{L^2}^2 \left\| \mathcal{L}^{\frac{1}{2}} (\mathbb{S}_N \phi - \phi) \right\|_{L^2([0,t]; L^2(\mathbb{R}^3))} \\ & \quad + C \|\mathbb{S}_N g - g\|_{L^\infty([0,t]; L^2)} \|g_0\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} \phi\|_{L^2([0,t]; L^2)} \\ & \quad + C \|g_0\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} (\mathbb{S}_N g - g)\|_{L^2([0,t]; L^2)} \|\mathcal{L}^{\frac{1}{2}} \phi\|_{L^2([0,t]; L^2)}. \end{aligned}$$

Let  $N \rightarrow +\infty$  in (4.4), we conclude that, for any  $\phi(t) \in C^1(\mathbb{R}_+, \mathcal{S}(\mathbb{R}^3))$ ,

$$\begin{aligned} & (g(t), \phi(t))_{L^2(\mathbb{R}^3)} - (g(0), \phi(0))_{L^2(\mathbb{R}^3)} \\ & = - \int_0^t (\mathcal{L}g(\tau), \phi(\tau))_{L^2(\mathbb{R}^3)} d\tau + \int_0^t (\Gamma(g(\tau), g(\tau)), \phi(\tau))_{L^2(\mathbb{R}^3)} d\tau, \end{aligned}$$

which shows  $g \in L^\infty([0, +\infty[; L^2(\mathbb{R}^3))$  is a global weak solution of Cauchy problem (1.3).

**4.3. Regularity of the solution.** For  $\mathbb{S}_N g$  defined in (4.1), since

$$\lambda_{n,0} \geq \lambda_{2,0} > 0, \quad \forall n \geq 2,$$

we deduce from the formulas (4.3) and the orthogonality of the basis  $(\varphi_{n,0,0})_{n \in \mathbb{N}}$  that

$$\begin{aligned} & \frac{d}{dt} \|e^{\frac{t}{2}\mathcal{L}} \mathbb{S}_N g(t)\|_{L^2}^2 + \frac{\lambda_{2,0}}{2} \|e^{\frac{t}{2}\mathcal{L}} \mathbb{S}_N g\|_{L^2}^2 \\ & \leq \frac{d}{dt} \|e^{\frac{t}{2}\mathcal{L}} \mathbb{S}_N g(t)\|_{L^2}^2 + \frac{1}{2} \sum_{n=2}^N e^{\lambda_{n,0}t} \lambda_{n,0} |g_n(t)|^2 \\ & = \frac{d}{dt} \|e^{\frac{t}{2}\mathcal{L}} \mathbb{S}_N g(t)\|_{L^2}^2 + \frac{1}{2} \|e^{\frac{t}{2}\mathcal{L}} \mathcal{L}^{\frac{1}{2}} \mathbb{S}_N g\|_{L^2}^2 \leq 0. \end{aligned}$$

We have then

$$\frac{d}{dt} \left( e^{-\frac{\lambda_{2,0}t}{2}} \|e^{\frac{t}{2}\mathcal{L}} \mathbb{S}_N g(t)\|_{L^2}^2 \right) \leq 0,$$

it deduces that for any  $t > 0$ , and  $N \in \mathbb{N}$ ,

$$\|e^{\frac{t}{2}\mathcal{L}} \mathbb{S}_N g(t)\|_{L^2(\mathbb{R}^3)} \leq e^{-\frac{\lambda_{2,0}t}{4}} \|g_0\|_{L^2(\mathbb{R}^3)}.$$

By using the monotone convergence theorem and the formula (2.3), we conclude that, there exists a constant  $c_0 > 0$ , such that

$$(4.6) \quad \|e^{c_0 t (\log(\mathcal{H}+1))^{\frac{2}{s}}} g(t)\|_{L^2(\mathbb{R}^3)} \leq e^{-\frac{\lambda_{2,0}t}{4}} \|g_0\|_{L^2(\mathbb{R}^3)}.$$

This is the formula (1.5).

For the case 1) when  $0 < s \leq 2$ , the orthogonality of the basis  $(\varphi_{n,0,0})_{n \in \mathbb{N}}$  implies that,

$$\begin{aligned} \|g\|_{Q^{2c_0t}(\mathbb{R}^3)} &= \|(\mathcal{H} + 1)^{c_0t} g\|_{L^2(\mathbb{R}^3)} = \sum_{n=0}^{+\infty} (2n + \frac{5}{2})^{c_0t} |g_n|^2 \\ &\leq \sum_{n=0}^{+\infty} e^{c_0t (\log(2n + \frac{5}{2}))^{\frac{2}{s}}} |g_n|^2 = \|e^{c_0t (\log(\mathcal{H}+1))^{\frac{2}{s}}} g(t)\|_{L^2(\mathbb{R}^3)}^2 \\ &\leq e^{-\frac{\lambda_{2,0}t}{4}} \|g_0\|_{L^2(\mathbb{R}^3)}^2. \end{aligned}$$

This is the formula (1.6).

For the part 2) of Theorem 1.1, in case  $0 < s < 2$ , we deduce from Proposition 6.1 and formula (4.6), the formula (1.7) follows.

The proof of Theorem 1.1 is completed.

## 5. ESTIMATE ON THE NONNEAR EIGENVALUE

In this section, we provide the proof of Lemma 3.2.

**Lemma 5.1.** *For  $n \geq 2$  and  $\mu_{k,l}$  was defined in (2.2) with  $k, l \in \mathbb{N}$ ,  $0 < s \leq 2$ , we have*

$$(5.1) \quad \sum_{\substack{k+l=n \\ k \geq 1, l \geq 1}} \frac{|\mu_{k,l}|^2}{(\log(2l + \frac{5}{2}))^{2/s}} \lesssim (\log(2n + \frac{5}{2}))^{2/s}.$$

*Proof.* Recall from (2.2) that

$$\mu_{k,l} = \sqrt{\frac{(2k+2l+1)!}{(2k+1)!(2l+1)!}} \left( \int_0^{\frac{\pi}{2}} \beta(\theta) (\sin \theta)^{2k} (\cos \theta)^{2l} d\theta \right),$$

and Beta function (1.4) that  $\beta(\theta) \sim (\sin \theta)^{-1} (\log(\sin \theta)^{-1})^{\frac{2}{s}-1}$ , we obtain

$$|\mu_{k,l}|^2 \sim \frac{(2k+2l+1)!}{(2k+1)!(2l+1)!} \left( \int_0^{\frac{\pi}{2}} (\log(\sin \theta)^{-1})^{\frac{2}{s}-1} (\sin \theta)^{2k-1} (\cos \theta)^{2l} d\theta \right)^2.$$

By using the substitution rule with  $x = (\sin \theta)^2$ , then

$$\begin{aligned} & \int_0^{\frac{\pi}{4}} (\log(\sin \theta)^{-1})^{\frac{2}{s}-1} (\sin \theta)^{2k-1} (\cos \theta)^{2l} d\theta \\ &= 2^{-\frac{2}{s}} \int_0^{\frac{1}{2}} (\log x^{-1})^{\frac{2}{s}-1} x^{k-1} (1-x)^{l-\frac{1}{2}} dx. \end{aligned}$$

This shows that,

$$|\mu_{k,l}|^2 \sim \frac{(2k+2l+1)!}{(2k+1)!(2l+1)!} \left( \int_0^{\frac{1}{2}} \left( \log \frac{1}{x} \right)^{\frac{2}{s}-1} x^{k-1} (1-x)^{l-\frac{1}{2}} dx \right)^2.$$

Without loss of generality, we assume  $n \gg 1$ . We can divide the summation into two parts:

$$\begin{aligned} \sum_{\substack{k+l=n \\ k \geq 1, l \geq 1}} \frac{|\mu_{k,l}|^2}{(\log(2l + \frac{5}{2}))^{2/s}} &= \sum_{l=1}^{n-1} \frac{|\mu_{n-l,l}|^2}{(\log(2l + \frac{5}{2}))^{2/s}} \\ (5.2) \quad &= \frac{|\mu_{n-1,1}|^2}{(\log \frac{9}{2})^{2/s}} + \sum_{l=2}^{n-1} \frac{|\mu_{n-l,l}|^2}{(\log(2l + \frac{5}{2}))^{2/s}} = \mathbf{H} + \mathbf{I}. \end{aligned}$$

For the estimate of  $\mathbf{H}$ , since  $1-x \sim 1$ , we have

$$\begin{aligned} |\mu_{n-1,1}|^2 &\sim \frac{(2n+1)!}{(2n-1)!3!} \left( \int_0^{\frac{1}{2}} \left( \log \frac{1}{x} \right)^{\frac{2}{s}-1} x^{n-2} dx \right)^2 \\ &= \frac{(2n+1)!}{(2n-1)!3!} \left( \int_{\log 2}^{+\infty} x^{\frac{2}{s}-1} e^{-(n-1)x} dx \right)^2 \\ &= \frac{(2n+1)!}{(2n-1)!3!} \frac{1}{(n-1)^{4/s}} \left( \int_{(n-1)\log 2}^{+\infty} u^{\frac{2}{s}-1} e^{-u} du \right)^2. \end{aligned}$$

It follows from the condition  $0 < s \leq 2$  that,

$$(5.3) \quad \mathbf{H} = \frac{|\mu_{n-1,1}|^2}{(\log \frac{9}{2})^{2/s}} \lesssim \frac{2(2n+1)n}{n^{4/s}4^{n-1}} \lesssim 1.$$

We now estimate  $\mathbf{I}$ . By using the inequality

$$\left( \int_0^{\frac{1}{l}} + \int_{\frac{1}{l}}^{\frac{1}{l}} \right)^2 \leq 2 \left( \int_0^{\frac{1}{l}} \right)^2 + 2 \left( \int_{\frac{1}{l}}^{\frac{1}{l}} \right)^2,$$

one can verify that

$$\begin{aligned} \mathbf{I} &= \sum_{l=2}^{n-1} \frac{1}{(\log(2l + \frac{5}{2}))^{2/s}} \frac{(2n+1)!}{(2n-2l+1)!(2l+1)!} \left( \int_0^{\frac{1}{2}} (\log x^{-1})^{\frac{2}{s}-1} x^{n-l-1} (1-x)^l dx \right)^2 \\ &\leq \sum_{l=2}^{n-1} \frac{2}{(\log(2l + \frac{5}{2}))^{2/s}} \frac{(2n+1)!}{(2n-2l+1)!(2l+1)!} \left( \int_{\frac{1}{l}}^{\frac{1}{2}} (\log x^{-1})^{\frac{2}{s}-1} x^{n-l-1} (1-x)^l dx \right)^2 \\ &\quad + \sum_{l=2}^{n-1} \frac{2}{(\log(2l + \frac{5}{2}))^{2/s}} \frac{(2n+1)!}{(2n-2l+1)!(2l+1)!} \left( \int_0^{\frac{1}{l}} (\log x^{-1})^{\frac{2}{s}-1} x^{n-l-1} (1-x)^l dx \right)^2 \\ (5.4) \quad &= \mathbf{I}_1 + \mathbf{I}_2. \end{aligned}$$



For the part  $\mathbf{I}_1$ . It is obviously that

$$\begin{aligned}
& \left( \int_{\frac{1}{7}}^{\frac{1}{2}} (\log x^{-1})^{\frac{2}{s}-1} x^{n-l-1} (1-x)^l dx \right)^2 \\
& \lesssim (\log l)^{4/s} \left( \int_{\frac{1}{7}}^{\frac{1}{2}} (\log(1/x))^{-1} x^{n-l-1} (1-x)^{l+\frac{1}{2}} dx \right)^2 \\
& \leq (\log l)^{4/s} \left( \int_{\frac{1}{7}}^{\frac{1}{2}} (\log(1/x))^{-2} x^{-1} dx \right) \left( \int_0^1 x^{2n-2l-1} (1-x)^{2l+1} dx \right) \\
& = \frac{(2n-2l-1)!(2l+1)!(\log l)^{4/s}}{(2n+1)!} \left( \int_{\log 2}^{\log l} x^{-2} dx \right) \\
& \lesssim \frac{(2n-2l-1)!(2l+1)!(\log l)^{4/s}}{(2n+1)!}.
\end{aligned}$$

We obtain

$$\begin{aligned}
\mathbf{I}_1 & \lesssim \sum_{l=2}^{n-1} \frac{1}{(\log(2l + \frac{5}{2}))^{2/s}} \frac{(2n+1)!}{(2n-2l+1)!(2l+1)!} \frac{(2n-2l-1)!(2l+1)!(\log l)^{4/s}}{(2n+1)!} \\
& \leq \left( \log(2n + \frac{5}{2}) \right)^{2/s} \sum_{l=2}^{n-1} \frac{1}{(2n-2l+1)(2n-2l)} \\
& = \left( \log(2n + \frac{5}{2}) \right)^{2/s} \sum_{k=1}^{n-2} \frac{1}{(2k+1)(2k)} \\
& \leq \left( \log(2n + \frac{5}{2}) \right)^{2/s}.
\end{aligned}$$

For the part  $\mathbf{I}_2$ , we divide the summation into two parts,

$$\begin{aligned}
\mathbf{I}_2 & = \sum_{l=2}^{\leq \frac{n}{2}} \frac{1}{(\log(2l + \frac{5}{2}))^{2/s}} \frac{(2n+1)!}{(2n-2l+1)!(2l+1)!} \left( \int_0^{\frac{1}{7}} (\log x^{-1})^{\frac{2}{s}-1} x^{n-l-1} (1-x)^l dx \right)^2 \\
& \quad + \sum_{\frac{n}{2} < l \leq n-1} \frac{1}{(\log(2l + \frac{5}{2}))^{2/s}} \frac{(2n+1)!}{(2n-2l+1)!(2l+1)!} \left( \int_0^{\frac{1}{7}} (\log x^{-1})^{\frac{2}{s}-1} x^{n-l-1} (1-x)^l dx \right)^2 \\
& = \mathbf{I}_{21} + \mathbf{I}_{22}.
\end{aligned}$$

For the estimation of  $\mathbf{I}_{21}$ : Transforming  $x = \frac{u}{l}$ , we have

$$(5.5) \quad \int_0^{\frac{1}{7}} (\log x^{-1})^{\frac{2}{s}-1} x^{n-l-1} (1-x)^l dx = \int_0^1 \left( \log \frac{l}{u} \right)^{\frac{2}{s}-1} \frac{u^{n-l-1}}{l^{n-l}} (1 - \frac{u}{l})^l du.$$

Since

$$\frac{1}{l^{n-l}} \begin{cases} = 4 \frac{1}{2^n}, & l = 2; \\ \leq 9 \frac{1}{2^n}, & l = 3; \\ \leq \frac{1}{4^{\frac{n}{2}}} = \frac{1}{2^n}, & 4 \leq l \leq \frac{n}{2}, \end{cases}$$

we conclude that, for  $2 \leq l \leq \frac{n}{2}$ ,

$$\frac{1}{l^{n-l}} \lesssim \frac{1}{2^n}.$$

At the same time, since  $(1 + \frac{1}{l})^l$  is increasing with respect to  $l$ , we obtain, for  $0 < u < 1$ ,

$$1 > (1 - \frac{u}{l})^l > (1 - \frac{1}{l})^l = \frac{1 - \frac{1}{l}}{(1 + \frac{1}{l-1})^{l-1}} \geq \frac{1}{4}.$$

It follows from the estimation (5.5) that

$$\begin{aligned} & \int_0^{\frac{1}{l}} (\log x^{-1})^{\frac{2}{s}-1} x^{n-l-1} (1-x)^l dx \\ & \lesssim \frac{1}{2^n} \int_0^1 \left( \log \frac{l}{u} \right)^{\frac{2}{s}-1} u^{n-l-1} du \lesssim \frac{1}{2^n} (\log l)^{\frac{2}{s}-1}. \end{aligned}$$

Therefore, we can estimate

$$\begin{aligned} \mathbf{I}_{21} & \lesssim \sum_{l=2}^{\leq \frac{n}{2}} \frac{1}{(\log(2l + \frac{5}{2}))^{2/s}} \frac{(2n+1)!}{(2n-2l+1)!(2l+1)!} \left( \frac{1}{2^n} (\log l)^{\frac{2}{s}-1} \right)^2 \\ & \lesssim (\log(2n + \frac{5}{2}))^{2/s} \sum_{l=2}^{\leq \frac{n}{2}} \frac{(2n)!}{(2n-2l)!(2l)!} \frac{1}{2^{2n}} \leq (\log(2n + \frac{5}{2}))^{2/s}. \end{aligned}$$

For the part  $\mathbf{I}_{22}$ ,

$$\begin{aligned} \mathbf{I}_{22} & \lesssim \frac{1}{(\log(2n + \frac{5}{2}))^{2/s}} \sum_{\frac{n}{2} < l \leq n-1} \frac{(2n)!}{(2n-2l)!(2l)!} \left( \int_0^{\frac{n}{l}} \left( \log \frac{n}{u} \right)^{\frac{2}{s}-1} \frac{u^{n-l-1}}{n^{n-l}} \left( 1 - \frac{u}{n} \right)^l du \right)^2 \\ & \leq \frac{1}{(\log(2n + \frac{5}{2}))^{2/s}} \left( \int_0^2 \left( \log \frac{n}{u} \right)^{\frac{4}{s}-2} du \right) \\ & \quad \times \left( \int_0^2 \sum_{\frac{n}{2} < l \leq n-1} \frac{(2n)!}{(2n-2l)!(2l)!} \frac{u^{2n-2l}}{n^{2n-2l}} \left( 1 - \frac{u}{n} \right)^{2l} \frac{du}{u^2} \right). \end{aligned}$$

For the first integral,

$$\begin{aligned} \int_0^2 \left( \log \frac{n}{u} \right)^{\frac{4}{s}-2} du & \lesssim \int_0^2 (\log n)^{\frac{4}{s}-2} du + \int_0^2 \left( \log \frac{1}{u} \right)^{\frac{4}{s}-2} du \\ & \lesssim (\log n)^{\frac{4}{s}-2} + \int_{\log 2}^{+\infty} x^{\frac{4}{s}-2} e^{-x} dx \lesssim (\log n)^{\frac{4}{s}-2}. \end{aligned}$$

We now consider the second integral. Since for  $0 < u < 2$ ,

$$\begin{aligned} \sum_{\frac{n}{2} < l \leq n-1} \frac{(2n)!}{(2n-2l)!(2l)!} \frac{u^{2n-2l}}{n^{2n-2l}} \left( 1 - \frac{u}{n} \right)^{2l} & \leq \sum_{m=0}^{2n-2} \frac{(2n)!}{(2n-m)!m!} \frac{u^{2n-m}}{n^{2n-m}} \left( 1 - \frac{u}{n} \right)^m \\ & = 1 - \left( 1 - \frac{u}{n} \right)^{2n} - 2n \frac{u}{n} \left( 1 - \frac{u}{n} \right)^{2n-1}. \end{aligned}$$

Using the Taylor formula for  $f(x) = 1 - (1-x)^{2n} - 2nx(1-x)^{2n-1}$  with  $0 < x < 1$ ,

$$f(x) = f(0) + f'(\theta x)x = 2n(2n-1)\theta(1-\theta x)^{2n-2}x^2, \text{ for some } 0 < \theta < 1,$$

we obtain, for  $0 < u < 2, n \geq 2$ ,

$$\sum_{\frac{n}{2} < l \leq n-1} \frac{(2n)!}{(2n-2l)!(2l)!} \frac{u^{2n-2l}}{n^{2n-2l}} \left(1 - \frac{u}{n}\right)^{2l} \leq 4u^2.$$

This shows that

$$\left( \int_0^2 \sum_{\frac{n}{2} < l \leq n-1} \frac{(2n)!}{(2n-2l)!(2l)!} \frac{u^{2n-2l}}{n^{2n-2l}} \left(1 - \frac{u}{n}\right)^{2l} \frac{du}{u^2} \right) \leq 8.$$

Therefore, we have

$$\mathbf{I}_{22} \lesssim \left( \log(2n + \frac{5}{2}) \right)^{\frac{2}{s}}.$$

Combining with the estimate of  $\mathbf{I}_1, \mathbf{I}_{21}, \mathbf{I}_{22}$  and substituting it into (5.4), we have

$$\mathbf{I} = \sum_{l=2}^{n-1} \frac{|\mu_{n-1,l}|^2}{(\log(2l + \frac{5}{2}))^{2/s}} \leq (\log(2n + \frac{5}{2}))^{2/s}.$$

Reminding the estimate of  $\mathbf{H}$  in (5.3), the formula (5.1) follows. This end the proof of Lemma 5.1.  $\square$

## 6. APPENDIX

The important known results but really needed for this paper are presented in this section. For the self-content of paper, we will present some proof of those properties.

**6.1. Shubin spaces.** We refer the reader to the works [11], [27] for the Shubin spaces. Let  $\tau \in \mathbb{R}$ , The Shubin spaces  $Q^\tau(\mathbb{R}^3)$  can be also characterized through the decomposition into the Hermite basis:

$$\begin{aligned} f \in Q^\tau(\mathbb{R}^3) &\Leftrightarrow f \in L^2(\mathbb{R}^3), \left\| (\mathcal{H} + 1)^{\frac{\tau}{2}} f \right\|_{L^2} < +\infty; \\ &\Leftrightarrow f \in L^2(\mathbb{R}^3), \left\| \left( (|\alpha| + \frac{5}{2})^{\tau/2} (f, H_\alpha)_{L^2} \right)_{\alpha \in \mathbb{N}^3} \right\|_{l^2} < +\infty, \end{aligned}$$

where  $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ ,

$$H_\alpha(v) = H_{\alpha_1}(v_1) H_{\alpha_2}(v_2) H_{\alpha_3}(v_3), \quad \alpha \in \mathbb{N}^3,$$

and for  $x \in \mathbb{R}, n \in \mathbb{N}$ ,

$$H_n(x) = \frac{1}{(2\pi)^{\frac{1}{4}}} \frac{1}{\sqrt{n!}} \left( \frac{x}{2} - \frac{d}{dx} \right)^n (e^{-\frac{x^2}{4}}).$$

The following proof is based on the Appendix in [20].

*Proof.* Setting

$$A_{\pm,j} = \frac{v_j}{2} \mp \frac{d}{dv_j}, \quad j = 1, 2, 3,$$

we have, for  $\alpha \in \mathbb{N}^3, v \in \mathbb{R}^3$ ,

$$H_\alpha(v) = \frac{1}{\sqrt{\alpha_1! \alpha_2! \alpha_3!}} A_{+,1}^{\alpha_1} H_0(v_1) A_{+,2}^{\alpha_2} H_0(v_2) A_{+,3}^{\alpha_3} H_0(v_3),$$

and for  $j=1,2,3$ ,

$$A_{+,j} H_\alpha = \sqrt{\alpha_j + 1} H_{\alpha+e_j}, \quad A_{-,j} H_\alpha = \sqrt{\alpha_j} H_{\alpha-e_j} (= 0 \text{ if } \alpha_j = 0)$$

where  $(e_1, e_2, e_3)$  stands for the canonical basis of  $\mathbb{R}^3$ . For the harmonic oscillator  $\mathcal{H} = -\Delta + \frac{|v|^2}{4}$  of 3-dimension and  $s > 0$ , one can verify that,

$$\mathcal{H} = \frac{1}{2} \sum_{j=1}^3 (A_{+,j} A_{-,j} + A_{-,j} A_{+,j}).$$

Therefore, we have

$$\begin{aligned} \mathcal{H} H_\alpha &= \frac{1}{2} \sum_{j=1}^3 (A_{+,j} A_{-,j} + A_{-,j} A_{+,j}) H_\alpha \\ &= \frac{1}{2} \left[ \sum_{j=1}^3 \sqrt{\alpha_j} A_{+,j} H_{\alpha-e_j} + \sum_{j=1}^3 \sqrt{\alpha_j + 1} A_{-,j} H_{\alpha+e_j} \right] \\ &= \frac{1}{2} \sum_{j=1}^3 (2\alpha_j + 1) H_\alpha = \sum_{j=1}^3 \left( \alpha_j + \frac{1}{2} \right) H_\alpha. \end{aligned}$$

By using this spectral decomposition, we conclude that

$$(\mathcal{H} + 1)^{\frac{s}{2}} H_\alpha = (\lambda_\alpha + 1)^{\frac{s}{2}} H_\alpha, \quad \lambda_\alpha = \sum_{j=1}^3 \left( \alpha_j + \frac{1}{2} \right), \quad \alpha \in \mathbb{N}^3.$$

This ends the proof of the another definition of the Shubin space.  $\square$

**6.2. Smoothing effects.** Concerning the Shubin spaces introduced in part 6.1, we have the following property :

**Proposition 6.1.** *Let  $0 < s < 2$  and  $\tau > 0$ . There exists a constant  $C = C_s$  such that,*

$$\forall k \geq 1, \quad \|f\|_{Q^k(\mathbb{R}^3)} = \left\| \left( \mathcal{H} + 1 \right)^{\frac{k}{2}} f \right\|_{L^2} \leq e^{C \left( \frac{1}{\tau} \right)^{\frac{s}{2-s}} k^{\frac{2}{2-s}}} \|e^{\tau(\log(\mathcal{H}+1)) \frac{s}{2}} f\|_{L^2(\mathbb{R}^3)}$$

where  $\mathcal{H} = -\Delta + \frac{|v|^2}{4}$ .

*Proof.* Expanding  $f$  in the Hermite basis, and noting  $f_\alpha = (f, H_\alpha)_{L^2}$  as in Subsection 6.1, we get

$$\sum_{\alpha} e^{2\tau(\log(1+\lambda_\alpha)) \frac{s}{2}} |f_\alpha|^2 = \|e^{\tau(\log(\mathcal{H}+1)) \frac{s}{2}} f\|_{L^2(\mathbb{R}^3)}^2.$$

We rephrase the previous identity as follows

$$\sum_{\alpha \in \mathbb{N}^3} [h_{\tau,k}(1 + \lambda_\alpha)] (1 + \lambda_\alpha)^k |f_\alpha|^2 = \|e^{\tau(\log(\mathcal{H}+1)) \frac{s}{2}} f\|_{L^2(\mathbb{R}^3)}^2$$

where  $h_{\tau,k}(x) = \frac{e^{2\tau(\log x) \frac{s}{2}}}{x^k}$ . It is easy to check that

$$(6.1) \quad \forall x \geq 1, \quad h_{\tau,k}(x) \geq e^{-\frac{2-s}{2} \left( \frac{s}{4\tau} \right)^{\frac{s}{2-s}} k^{\frac{2}{2-s}}}.$$

Indeed, for  $0 < s < 2$ , using Young's inequality

$$xy \leq \frac{1}{p} x^p + \frac{1}{q} y^q, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1,$$

with  $p = \frac{2}{2-s}$ ,  $q = \frac{2}{s}$ , we obtain

$$k \log x \leq \frac{2-s}{2} \left[ \left( \frac{s}{4\tau} \right)^{\frac{s}{2-s}} k \right]^{\frac{2}{2-s}} + 2\tau(\log x)^{\frac{2}{s}}.$$

Therefore,

$$h_{\tau,k}(x) = e^{2\tau(\log x)^{\frac{s}{2}} - k \log x} \geq e^{-\frac{2-s}{2} \left[ \left( \frac{s}{4\tau} \right)^{\frac{s}{2}} k \right]^{\frac{2}{2-s}}}.$$

Then (6.1) follows immediately. We conclude that

$$\begin{aligned} \|e^{\tau(\log(\mathcal{H}+1))^{\frac{s}{2}}} f\|_{L^2(\mathbb{R}^3)}^2 &= \sum_{\alpha \in \mathbb{N}^3} h_{\tau,k}(1 + \lambda_{\alpha})^k |f_{\alpha}|^2 \\ &\geq e^{-\frac{2-s}{2} \left( \frac{s}{4\tau} \right)^{\frac{s}{2}} k^{\frac{2}{2-s}}} \sum_{\alpha \in \mathbb{N}^3} (1 + \lambda_{\alpha})^k |f_{\alpha}|^2 = e^{-C_s \left( \frac{1}{\tau} \right)^{\frac{s}{2-s}} k^{\frac{2}{2-s}}} \|f\|_{Q^k(\mathbb{R}^3)}^2, \end{aligned}$$

where we used the result of Subsection 6.1. This ends the proof.  $\square$

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